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ON SOLITONS IN FERMI–PASTA–ULAM-TYPE SYSTEMS

Abstract. *The article studies Fermi-Pasta-Ulam-type systems, which describe the dynamics of systems of nonlinearly coupled particles. A brief review of known results on the existence of solitary traveling waves in such systems is given.*

Keywords: *Fermi–Pasta–Ulam-type systems, solitons, solitary traveling waves, periodic traveling waves, critical point theory.*

In 1953, one of the most prominent physicists of the twentieth century E. Fermi asked his colleagues at the Los Alamos Laboratory S. Ulam, J. Pasta and M. Tsingou to investigate the problem of thermalization of energy in nonlinear discretely loaded strings using the example of oscillations of 64 masses connected to each other by springs (fig. 1).

Fig. 1

By creating the initial oscillation, the researchers wanted to see how this initial vibrational mode would be distributed across all other modes. It was assumed that the energy would eventually be uniformly distributed between the modes, i.e. along the entire wavelength, thus, the thermalization of energy will occur. In fact, the problem was reduced to the study of the behavior of systems of ordinary differential equations, which were initially linear, but in which nonlinearity was introduced as perturbation. If there were no such perturbation, then the energy of each normal mode of a linear system, i.e. oscillations with a given frequency, would be constant. Therefore, it can be expected that nonlinear interactions between modes will lead to the fact that the energy of the system is uniformly distributed between the modes.

The motion of the chain, which they considered, was described by the system

$$m\ddot{q}_n(t) = U'\big(q_{n+1}(t) - q_n(t)\big) - U'\big(q_n(t) - q_{n-1}(t)\big),\ n = \overline{1, N}, \tag{1}$$

where $q_n(t)$ is the displacement from the equilibrium position (coordinate) of the n-th mass at time t, $N = 64$. They considered potentials of the form

$$U(r) = \frac{\gamma}{2}r^2 + \frac{\alpha}{3}r^3,\ U(r) = \frac{\gamma}{2}r^2 + \frac{\beta}{4}r^4. \quad (2)$$

In 1954, after the calculations of the FPU problem on the computer "MANIAC I" (Mathematical Analyzer, Numerical Integrator and Computer), they did not receive the expected result. They found that the transfer of energy into two or three modes does occur at the initial stage of the calculation, but then a return to the initial state is observed. [1]

Several mathematicians and physicists have learned about this paradox of the return of the initial oscillation. In particular, American physicists N. Zabuski and M. Kruskal decided to continue computational experiments with the model proposed by Fermi. It turned out that special solitary waves appear in the chain, which prevent the energy from being uniformly distributed along its entire length. In 1965, N. Zabuski and M. Kruskal established that the FPU equation, when the distance between the weights is reduced and their number increases indefinitely, transitions into the KdV equation, whose solution is the solitary wave. One remarkable property of these waves is that after interacting, their shape and speed are restored (fig. 2).

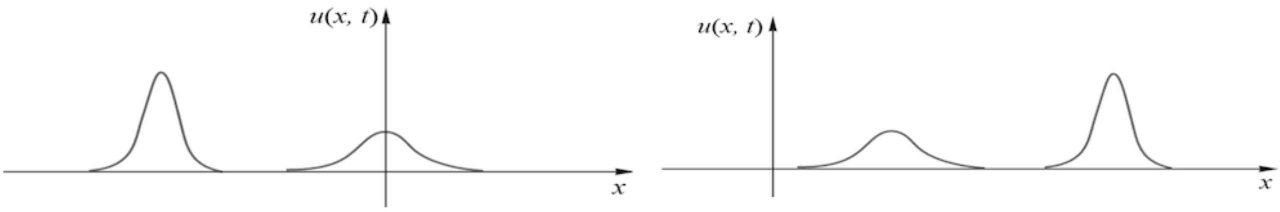

Fig. 2. Solitary waves before and after interaction

This process resembles an elastic collision between two particles. Therefore, Kruskal and Zabuski named these solitary waves "*solitons*". This special term for solitary waves, which is similar to the names of elementary particles like the electron, proton, and many others, is now widely accepted. [2]

It is worth noting that relatively recently (in 2015) a group of scientists led by the American scientist Yu. Lvov managed to find a solution to the Fermi-Pasta-Ulam problem. Their article provides a mathematical explanation of what level of energy is required to create one complete wave in a chain of connected masses that strive for thermal equilibrium. [3]

The works of E. Fermi, J. Pasta and S. Ulam motivated a large number of further numerous and analytical investigations. We mention here the so-called Toda lattice which is a completely integrable system. Unfortunately, the Toda lattice is the only known completely integrable lattice of FPU-type. [4]

One of the first rigorous results about general FPU-type lattices was obtained by G. Friesecke and J. Wattis ([5]). They have proved the existence of solitary travelling waves in monoatomic FPU lattices under some general assumptions on the potential of interparticle interaction. The approach of G. Friesecke and J. Wattis is based on an appropriate constrained minimization method and the concentration compactness principle of P.-L. Lions.

A. Pankov and R. Pflüger ([6]) revised the last approach considerably choosing periodic travelling waves as a starting point. The existence of periodic waves is obtained by means of the standard mountain pass theorem. Then one gets solitary waves in the limit as the wave lengths (period) goes to infinity (periodic approximations method).

Similar problem for infinite lattices was studied by G. Arioli, J. Chabrowski, F. Gazzola, A. Szulkin, S. Terracini and others.

The review of the available results concerning such systems is given in [8].

I. Butt and J. Wattis ([7]) showed that a two-dimensional FPU lattice can be used to simulate the transfer of electric charge. This lattice comprises a square configuration of repeating units, each made up of two linear inductors and a nonlinear capacitor (fig. 3).

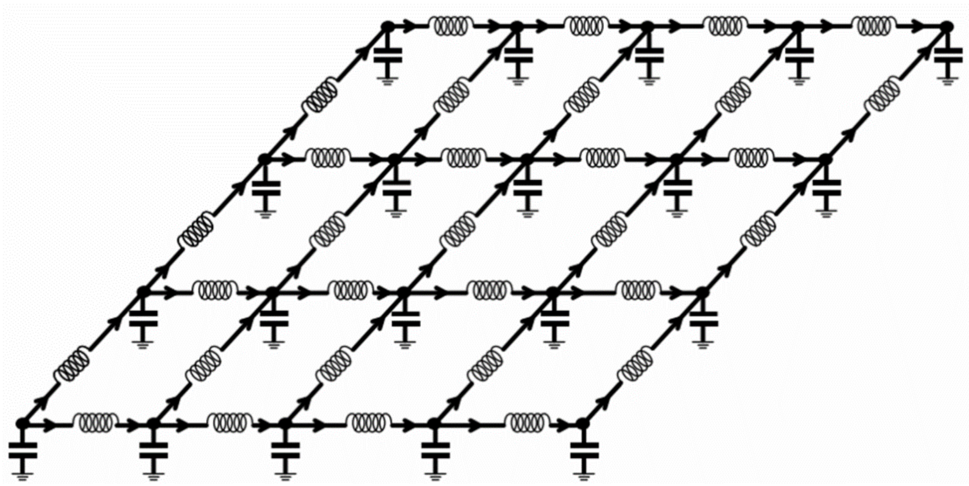

Fig. 3

We consider a Fermi–Pasta–Ulam-type system that describes the dynamics of an infinite system of nonlinearly coupled particles on a two-dimensional Butt–Wattis-type lattice with local interaction

$$\begin{gathered}\ddot{q}_{n,\,m}(t)=W_1'\big(q_{n+1,\,m}(t)-q_{n,\,m}(t)\big)-W_1'\big(q_{n,\,m}(t)-q_{n-1,\,m}(t)\big)+\\ +W_2'\big(q_{n,\,m+1}(t)-q_{n,\,m}(t)\big)-W_2'\big(q_{n,\,m}(t)-q_{n,\,m-1}(t)\big),\ (n,\,m)\in\mathbb{Z}^2,\end{gathered} \tag{3}$$

where $q_{n,m}(t)$ is a coordinate of (n,m)-th particle at time t, $W_1, W_2 \in C^1(\mathbb{R};\mathbb{R})$ are the potentials of neighbor interactions.

Traveling wave is a solution of the form

$$q_{n,\,m}(t)=u(n\cos\varphi+m\sin\varphi-ct), \tag{4}$$

where the function $u(s)$ is called the profile function, or simply profile, of the wave. Note that the vector $\vec{l}\,(\cos\varphi, \sin\varphi)$ defines the direction of wave propagation. The constant $c\neq 0$ is called the speed of the wave.

Using the critical point theory, we have studied the sufficient conditions for the existence of periodic and solitary traveling waves in such systems ([9]–[12]).

Now, we will consider the FPU-type system with nonlocal interaction, i.e., each particle interacts with l neighbors horizontally and vertically on each side. The equations of motion of the system considered are of the form

$$\ddot{q}_{n,m}(t)=\sum_{j=1}^{l}\Big[W_{1j}'\big(q_{n+j,m}(t)-q_{n,m}(t)\big)-W_{1j}'\big(q_{n,m}(t)-q_{n-j,m}(t)\big)+ \\ +W_{2j}'\big(q_{n,m+j}(t)-q_{n,m}(t)\big)-W_{2j}'\big(q_{n,m}(t)-q_{n,m-j}(t)\big)\Big],(n,m)\in\mathbb{Z}^2, \tag{5}$$

$W_{1j}, W_{2j}\in C^1(\mathbb{R};\mathbb{R})$ are the potentials of interaction ($j=1, 2,..., l$), in particular, W_{11} and W_{21} are the potentials of the horizontal and vertical interaction, respectively, of the (n,m)-th particle with nearest neighbors, W_{12} and W_{22} with second nearest neighbors, and so on.

Sufficient conditions for the existence of periodic traveling (see [13]) have been obtained using the critical point method and a suitable version of the Mountain Pass Theorem for functionals satisfying the Cerami condition instead of the Palais–Smale condition. And the existence of subsonic traveling waves ([14]), was established using the Linking Theorem instead of the Mountain Pass Theorem. In a sense, the case of solitary waves is a limit case of the periodic waves. Therefore, solitary waves were constructed by the periodic aproximations method (see [15]). Finally, an exponential estimate ([16]) for the profiles of such solutions is obtained by applying the Paley–Wiener Theorem.

In our next research, we plan to apply the constrained minimization method to the problem of the existence of traveling waves and prove the existence of ground waves in Fermi–Pasti–Ulam type systems.

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СОЛІТОНИ В СИСТЕМАХ ТИПУ ФЕРМІ–ПАСТИ–УЛАМА

Анотація. *У статті вивчаються системи типу Фермі-Пасти-Улама, які описують динаміку систем нелінійно звязаних частинок. Зроблено короткий огляд відомих результатів про існування відокремлених біжучих хвиль в таких системах.*

Ключові слова: *системи типу Фермі–Пасти–Улама, солітони, відокремлені біжучі хвилі, періодичні біжучі хвилі, теорія критичних точок.*