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ON TRAVELING WAVES IN FERMI–PASTA–ULAM-TYPE SYSTEMS WITH NONLOCAL INTERACTION ON A 2D-LATTICE

Abstract. We consider the Fermi–Pasta–Ulam-type systems that describe an infinite systems of particles with nonlocal interaction on a two dimensional lattice. The main result concerns the existence of periodic and solitary traveling waves solutions. By means of critical point theory, sufficient conditions for the existence of such solutions are obtained.

Key words and phrases: traveling waves, Fermi–Pasta–Ulam-type systems, 2D-lattice, critical point theory.

Discrete infinite-dimensional Hamiltonian systems are widely used for modeling complex quantum phenomena. Such systems include, in particular, infinite systems of nonlinear oscillators, discrete sine–Gordon-type equations, and Fermi–Pasta–Ulam-type systems. Such systems are of interest in view of numerous applications in physics. Among the solutions of such systems, traveling waves deserve special attention.

We study the Fermi–Pasta–Ulam-type system that describes an infinite system of nonlinearly coupled particles on a two dimensional lattice with nonlocal interaction, i.e., each particle interacts with l neighbors horizontally and vertically on each side. The equations of motion of the system considered are of the form

$$\begin{gathered}\ddot{q}_{n,m}(t)=\sum_{j=1}^{l}\Big[W_{1j}'\left(q_{n+j,m}(t)-q_{n,m}(t)\right)-W_{1j}'\left(q_{n,m}(t)-q_{n-j,m}(t)\right)+\\ +W_{2j}'\left(q_{n,m+j}(t)-q_{n,m}(t)\right)-W_{2j}'\left(q_{n,m}(t)-q_{n,m-j}(t)\right)\Big],(n,m)\in\mathbb{Z}^2,\end{gathered} \tag{1}$$

where $q_{n,m}(t)$ is the coordinate of the (n, m)-th particle at time t, $W_{1j}, W_{2j}\in C^1(\mathbb{R};\mathbb{R})$ are the potentials of interaction ($j=1, 2,..., l$).

In contrast to the previous results (see [1–6]), we study the Fermi–Pasta–Ulam-type systems with nonlocal interaction.

A traveling wave solution of Eq. (1) is a function of the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct), \tag{2}$$

where the profile function $u(s)$ of the wave, or simply profile, satisfies the equation

$$\begin{aligned} c^2u''(s) = \sum_{j=1}^{l}\Big[W'_{1j}\big(u(s+j\cos\varphi)-u(s)\big) - W'_{1j}\big(u(s)-u(s-j\cos\varphi)\big) + \\ +W'_{2j}\big(u(s+j\sin\varphi)-u(s)\big) + W'_{2j}\big(u(s)-u(s-j\sin\varphi)\big)\Big], \end{aligned} \tag{3}$$

where $s = n\cos\varphi + n\sin\varphi - ct.$

In what follows, a solution of Eq. (3) is understood as a function $u(s)$ from the space $C^2(\mathbb{R};\mathbb{R})$ satisfying Eq. (3) for all $s \in \mathbb{R}$.

We consider two types of solutions: periodic and solitary traveling waves. In the first case profile satisfies the following condition

$$u'(s+2k) = u'(s), \tag{4}$$

where $k > 0$ is a real number, i.e., the velocity profile $u'(s)$ is periodic. Note that the profile of such wave is not necessarily periodic. In the second case profile satisfies the following condition

$$\lim_{s\to\pm\infty} u'(s) = u'(\pm\infty) = 0, \tag{5}$$

i.e., the velocity profile $u'(s)$ vanishes at infinity.

We assume that

(i) $W_{ij}(r) = \frac{c_{ij}^2}{2}r^2 + f_{ij}(r)$, *where* $c_{ij} \in\in \mathbb{R}$, $f_{ij} \in C^1(\mathbb{R};\mathbb{R})$, $f_{ij}(0) = f'_{ij}(0) = 0$ *and* $f'_{ij}(r) = o(r)$ *as* $r \to 0$, $i = 1,2$, $j = 1, 2, ..., l$;

(ii) *all functions* $f_{ij}(r)$, $i = 1, 2$, $j = 1, 2, ..., l$, *are nonnegative, and there exists* l_0 *such that*

$$\lim_{r\to\pm\infty}\frac{f'_{il_0}(r)}{r^2} = +\infty;$$

In addition, if T := 2k is an integer period and φ ≡ 0, π/2 mod π (φ = πn or φ = π/2 + πn for any fixed n $\in \mathbb{Z}$*), then we assume that* l_0 *=1;*

(iii) *all functions* $|r|^{-1} f'_{ij}(r),\ i=1,2,\ j=1,2,...,l,$ *are nondecreasing, and there exists* l_0 *such that* $|r|^{-1} f'_{il_0}(r)$ *are strictly increasing.*

An important role is played by the quantity defined by the equality

$$c_0 = c_0(\varphi) := \sqrt{\sum_{j=1}^{l} \left(c_{1j}^2 \cos^2\varphi + c_{2j}^2 \sin^2\varphi\right) j^2}.$$

Let E_k be the Hilbert space defined by

$$E_k = \{u \in H^1_{loc}(\mathbb{R}) : u'(s+2k) = u'(s),\ u(0)=0\}$$

with the scalar product

$$(u,v)_k = \int_{-k}^{k} u'(s)v'(s)ds\,.$$

On E_k we introduce the functional

$$J_k(u) = \int_{-k}^{k} \left[\frac{c^2}{2}(u'(s))^2 - \sum_{j=1}^{l}\left(W_{1j}\left(A_j^+ u(s)\right) + W_{2j}\left(B_j^+ u(s)\right)\right)\right] ds,$$

where

$$\left(A_j^+ u\right)(s) := u(s + j\cos\varphi) - u(s)\,,$$

$$\left(B_j^+ u\right)(s) := u(s + j\sin\varphi) - u(s).$$

The critical points of the functional J_k are solutions of Eq. (3) satisfying (4). We obtain sufficient conditions for the existence of such solutions with the aid of critical point method and a suitable version of the Mountain Pass Theorem for functionals satisfying the Cerami condition instead of the Palais–Smale condition. We prove that under natural assumptions there exist monotone traveling waves with the speed $c > c_0$, i.e., there exist subsonic traveling waves.

Theorem 1 ([7]). *Assume* $(i)-(iii), c > c_0$ *and* $\varphi \in \left[\pi n, \frac{\pi}{2} + \pi n\right]$ *for any fixed* $n \in \mathbb{Z}$. *Then there exists* $T_0 \geq l$ *such that for every* $T := 2k \geq T_0$ *Eq. (3) has a nonconstant nondecreasing and nonincreasing solutions satisfying (4).*

Now we replace conditions (ii) and (iii) with the condition

(iv) *there exists* $\mu > 2$ *such that* $0 \le \mu f_{ij}(r) \le r f_{ij}'(r)$ *for* $r \ne 0,\ i = 1, 2,\ j = 1, 2, ..., l.$

In the next theorem we obtain the existence of traveling waves with the speed $0 < c \le c_0$, i.e., the existence of subsonic traveling waves with the aid of the Linking Theorem instead of the Mountain Pass Theorem.

Theorem 2 ([8]). *Assume* (i) *and* (iv). *Then for every* $c \in (0, c_0]$ *Eq. (3) has a nonconstant solution satisfying (4).*

Let E be the Hilbert space defined by

$$E = \{u \in H_{loc}^1(\mathbb{R}) : u' \in L^2(\mathbb{R}),\ u(0) = 0\}$$

with the scalar product

$$(u, v) = \int_{-\infty}^{+\infty} u'(s) v'(s) ds .$$

On E we introduce the functional

$$J(u) = \int_{-\infty}^{+\infty} \left[\frac{c^2}{2} (u'(s))^2 - \sum_{j=1}^{l} \Big(W_{1j}\left(A_j^+ u(s)\right) + W_{2j}\left(B_j^+ u(s)\right) \Big) \right] ds.$$

Any critical point of the functional J is a solution of Eq. (3) satisfying (5). In a sense, the case of solitary waves is a limit case of the periodic waves. Therefore, solitary waves were constructed by considering the critical points of the functional J_k and then passing to the limit as $k \to \infty$.

Theorem 3 ([9]). *Assume* $(i)-(iii), c > c_0$ *and* $\varphi \in \left[\pi n, \frac{\pi}{2} + \pi n \right]$ *for any fixed* $n \in \mathbb{Z}$.

Then Eq. (3) has a nonconstant nondecreasing and nonincreasing solutions satisfying (5).

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