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EXISTENCE OF SOLITARY TRAVELING WAVES IN A SYSTEM OF NONLINEARLY COUPLED OSCILLATORS ON THE 2D LATTICE

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We consider a system of differential equations that describes the dynamics of an infinite system of nonlinearly coupled nonlinear oscillators on the $2D$ lattice. By the method of critical points, we obtain a result on the existence of solitary traveling waves.

1. Introduction

In the present paper, we consider equations that describe the dynamics of an infinite system of nonlinearly coupled nonlinear oscillators on the two-dimensional integer lattice. Let $q_{n,m}(t)$ be a generalized coordinate of the (n,m)th oscillator at time t. Assume that each oscillator nonlinearly interacts with four its nearest neighbors. Then the equations of motion of the analyzed system take the form

$$\ddot{q}_{n,m} = U'(q_{n+1,m} - q_{n,m}) - U'(q_{n,m} - q_{n-1,m})$$
$$+ U'(q_{n,m+1} - q_{n,m}) - U'(q_{n,m} - q_{n,m-1}) - V'(q_{n,m}), \quad (n,m) \in \mathbb{Z}^2, \tag{1}$$

where $U, V \in C^1(\mathbb{R})$.

Equations (1) form an infinite system of ordinary differential equations. Moreover, for $V(r) \equiv 0$, this system is a two-dimensional analog of the Fermi–Pasta–Ulam system and, for

$$V(r) = K(1 - \cos r),$$

these equations represent a discrete $\sin$-Gordon equation on the 2D lattice.

Traveling waves form an important class of solutions for these systems. A fairly detailed presentation of the results on traveling waves in Fermi–Pasta–Ulam chains can be found in Pankov's works. Thus, in particular, the work [15] contains the most complete survey of the results of this kind. For the presentation of the results of investigation of these systems from the physical point of view, see [9]. Conditions for the existence of periodic traveling waves in the Fermi–Pasta–Ulam system on the 2D lattice were established in [6]. At the same time, the chains of oscillators were considered in several works. Thus, in particular, in [12], the results were obtained by the methods of bifurcation theory and, in [1, 8], the conditions for the existence of periodic and solitary traveling waves were established by the method of critical points. In [2, 5, 10, 11], traveling waves were studied for systems of linearly coupled oscillators on the 2D lattices. In particular, a system with odd 2π-periodic nonlinearity was investigated in [10] and the case of linear oscillators was considered in [11]. The conditions for the existence of periodic and solitary traveling waves were established in [2]. Periodic and homoclinic traveling waves for an infinite chain of nonlinearly connected nonlinear particles were studied in [14]. The existence of periodic

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traveling waves for the discrete sin-Gordon equation on the 2D lattice was established in [7]. The existence of subsonic periodic traveling waves in an infinite system of nonlinearly coupled nonlinear oscillators located on the 2D lattice was proved in [3]. The conditions for the existence of supersonic waves in this system were obtained in [4].

2. Statement of the Problem

Note that a traveling wave in the two-dimensional case has the form

$$q_{n,m}(t) = u(n\cos\varphi + m\sin\varphi - ct),$$

where the function of continuous argument $u(s)$ is called a profile and the constant $c \neq 0$ is called the velocity of traveling waves. For the profile $u(s)$, where

$$s = n\cos\varphi + m\sin\varphi - ct,$$

Eq. (1) takes the form

$$c^2u''(s) = U'(u(s+\cos\varphi) - u(s)) - U'(u(s) - u(s-\cos\varphi)) \\ + U'(u(s+\sin\varphi) - u(s)) - U'(u(s) - u(s-\sin\varphi)) - V'(u(s)). \tag{2}$$

Here and in what follows, a solution of Eq. (2) is defined as a function $u(s)$ from the class $C^2(\mathbb{R})$ satisfying Eq. (2) for all $s \in \mathbb{R}$.

Note that Eq. (2) contains the squared velocity c. This means that if the function $u(s)$ satisfies Eq. (2), then there exist two traveling waves with given profile u and velocities $\pm c$.

We are interested in solitary traveling waves whose profile satisfies the boundary condition

$$\lim_{s\to\pm\infty} u(s) = u(\pm\infty) = 0. \tag{3}$$

Note that these waves are also called homoclinic to zero.

3. Variational Statement of the Problem

By E we denote a Hilbert space $H^1(\mathbb{R})$ with the scalar product

$$(u,v) = \int\limits_{-\infty}^{+\infty} (u(s)v(s) + u'(s)v'(s))\,ds$$

and the corresponding norm

$$\|u\| = (u,u)^{\frac{1}{2}}.$$

Recall that, by the embedding theorem,

$$E \subset C_b(\mathbb{R}),$$

where $C_b(\mathbb{R})$ is the space of bounded continuous functions on $\mathbb{R}$. Moreover, the limit of functions from the space E at infinity is equal to zero.