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EXISTENCE OF HETEROCLINIC TRAVELING WAVES IN A SYSTEM OF OSCILLATORS ON A TWO-DIMENSIONAL LATTICE

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By using the method of critical points and the concentration-compactness principle, we study the problem of existence of heteroclinic traveling waves for a system of linearly coupled nonlinear oscillators on a two-dimensional lattice.

Introduction

Infinite-dimensional Hamiltonian systems are extensively used in nonlinear physics for the purposes of modeling of complex optical and quantum phenomena. In recent years, much attention has been given to models discrete in the space variable, such as the Frenkel–Kontorova model, the Fermi–Pasta–Ulam systems, the discrete nonlinear Schrödinger equations, the discrete sine-Gordon equations, chains of oscillators, etc. These systems are of interest from the viewpoint of numerous applications in physics [3, 5, 6].

An important class of solutions of these systems is formed by traveling waves. A detailed presentation of the results obtained for traveling waves in Fermi–Pasta–Ulam chains can be found in works by Pankov (see, e.g., the survey [12]). At the same time, there are only several works devoted to the investigation of traveling waves in the chains of oscillators. Among these works, we can mention the work [9] the results of which were obtained by the methods of theory of bifurcations and the works [1, 4] in which the conditions of existence of periodic and solitary traveling waves were obtained by the method of critical points.

In [13], periodic solutions for a system of oscillators located on two-dimensional lattices were studied, and in [2, 7, 8], traveling waves in these systems were investigated. In particular, in [7], a system with odd 2π-periodic nonlinearity was considered, and in [8], linear oscillators were investigated. The conditions of existence of periodic and solitary traveling waves were obtained in [2].

In [10], heteroclinic traveling waves were studied for the discrete sine-Gordon equation with linear interaction of the neighboring atoms in a one-dimensional lattice.

In the present paper, by using the method of critical points and the concentration-compactness principle, we investigate the problem of existence of heteroclinic traveling waves for the discrete sine-Gordon equation on a two-dimensional lattice.

The aim of the present paper is to establish conditions for the existence of heteroclinic traveling waves in a system of linearly coupled nonlinear oscillators with a potential

$$V(r) = K(1-\cos r)$$

in the two-dimensional integer lattice.

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Statement of the Problem

In the present paper, we study equations used for the description of the dynamics of an infinite system of linearly coupled nonlinear oscillators with a potential $V(r) = K(1-\cos r)$ located in a plane integer lattice. Let $q_{n,m}(t)$ be the generalized coordinate of the (n,m)th oscillator at time t. Assume that each oscillator linearly interacts with its four nearest neighbors. The equations of motion of this system have the form

$$\ddot{q}_{n,m}(t) = c_0^2(q_{n+1,m}(t) + q_{n-1,m}(t) + q_{n,m+1}(t) + q_{n,m-1}(t)$$

$$-4q_{n,m}(t)) - K\sin(q_{n,m}(t)), \qquad (n,m) \in \mathbb{Z}^2. \tag{1}$$

Equation (1) is an infinite system of ordinary differential equations and a two-dimensional analog of the discrete sine-Gordon equation with the linear interaction of neighboring atoms. It can also be represented in the form

$$\ddot{q}_{n,m} = c_0^2(\Delta q)_{n,m} - K\sin(q_{n,m}), \tag{2}$$

where

$$(\Delta q)_{n,m} = q_{n+1,m} + q_{n-1,m} + q_{n,m+1} + q_{n,m-1} - 4q_{n,m}$$

is a two-dimensional discrete Laplace operator.

It is worth noting that, in this case, the potential $V(r) = K(1-\cos r)$ does not satisfy the condition (h) from [1, 2] and, hence, it is impossible to obtain the results on the existence of periodic and solitary traveling waves in a similar way.

For the profile of traveling wave $u(s)$, where $s = n\cos\varphi + m\sin\varphi - ct$, $s \in \mathbb{R}$, Eq. (2) takes the form

$$c^2u''(s) = c_0^2(u(s+\cos\varphi) + u(s-\cos\varphi) + u(s+\sin\varphi) + u(s-\sin\varphi) - 4u(s)) - K\sin(u(s)), \tag{3}$$

where c is the velocity of propagation of the wave. Note that Eq. (3) contains solely the second power of the velocity c. Hence, if the function $u(s)$ satisfies Eq. (3), then there exist two traveling waves with this profile and velocities $\pm c$.

We consider the case of heteroclinic traveling waves. To determine the profile of these waves, it suffices to find the solution of Eq. (3) satisfying the conditions:

$$\lim_{s\to-\infty} u(s) = -\pi, \qquad \lim_{s\to+\infty} u(s) = \pi. \tag{4}$$

In what follows, the solution of Eq. (3) is understood as a function $u(s)$ from the class $C^2(\mathbb{R})$ satisfying Eq. (3) for all $s \in \mathbb{R}$.

For some values of the angle φ, problem (3), (4) can be reduced to the one-dimensional case, i.e., to a chain of oscillators (one-dimensional lattice).

For $\varphi = \frac{\pi k}{2}$, $k = 2n$, $n \in \mathbb{Z}$, we get

$$\cos\frac{\pi k}{2} = \cos\pi n = (-1)^n ,$$

$$\sin\frac{\pi k}{2} = \sin\pi n = 0 .$$

Then

$$c^2 u''(s) = c_0^2 (u(s+(-1)^n) + u(s-(-1)^n) + 2u(s) - 4u(s)) - K\sin(u(s))$$

$$= u(s+1) + u(s-1) - 2u(s) + c_0 u(s) - K\sin(u(s)).$$

For $\varphi = \frac{\pi k}{2}$, $k = 2n+1$, $n \in \mathbb{Z}$, we find

$$\cos\frac{\pi k}{2} = \cos\left(\frac{\pi}{2}(2n+1)\right) = 0 ,$$

$$\sin\frac{\pi k}{2} = \sin\left(\frac{\pi}{2}(2n+1)\right) = (-1)^n .$$

Then

$$c^2 u''(s) = c_0^2 (u(s) + u(s) + u(s+(-1)^n) + u(s-(-1)^n) - 4u(s)) - K\sin(u(s))$$

$$= c_0^2 (u(s+1) + u(s-1) - 2u(s)) - K\sin(u(s)).$$

For $\varphi = \frac{\pi}{4} + \frac{k\pi}{2}$, $k = 2n$, $n \in \mathbb{Z}$, we obtain

$$\cos\left(\frac{\pi}{4} + \frac{2n\pi}{2}\right) = \cos\left(\frac{\pi}{4} + \pi n\right) = (-1)^n \frac{\sqrt{2}}{2} ,$$

$$\sin\left(\frac{\pi}{4} + \frac{2n\pi}{2}\right) = \sin\left(\frac{\pi}{4} + \pi n\right) = (-1)^n \frac{\sqrt{2}}{2} .$$

Then

$$c^2 u''(s) = c_0^2 \left(2u\left(s + (-1)^n \frac{\sqrt{2}}{2}\right) + 2u\left(s - (-1)^n \frac{\sqrt{2}}{2}\right) - 4u(s)\right) - K\sin(u(s))$$

$$= c_0^2 \left(2u\left(s + \frac{\sqrt{2}}{2}\right) + 2u\left(s - \frac{\sqrt{2}}{2}\right) - 4u(s)\right) - K\sin(u(s)) .$$

Finally, for $\varphi = \frac{\pi}{4} + \frac{k\pi}{2}$, $k = 2n+1$, $n \in \mathbb{Z}$, we have

$$\cos\left(\frac{\pi}{4} + \frac{\pi k}{2}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{2}(2n+1)\right) = \cos\left(\frac{3\pi}{4} + \pi n\right) = (-1)^{n-1}\frac{\sqrt{2}}{2},$$

$$\sin\left(\frac{\pi}{4} + \frac{\pi k}{2}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{2}(2n+1)\right) = \sin\left(\frac{3\pi}{4} + \pi n\right) = (-1)^{n}\frac{\sqrt{2}}{2}.$$

Then

$$c^2 u''(s) = c_0^2\left(u\left(s + (-1)^{n-1}\frac{\sqrt{2}}{2}\right) + u\left(s - (-1)^{n-1}\frac{\sqrt{2}}{2}\right) + u\left(s + (-1)^{n}\frac{\sqrt{2}}{2}\right)\right.$$

$$\left. + u\left(s - (-1)^{n}\frac{\sqrt{2}}{2}\right) - 4u(s)\right) - K\sin(u(s))$$

$$= c_0^2\left(2u\left(s + \frac{\sqrt{2}}{2}\right) + 2u\left(s - \frac{\sqrt{2}}{2}\right) - 4u(s)\right) - K\sin(u(s)).$$

We perform the change of variables

$$\psi(s) = u\left(\frac{\sqrt{2}}{2}s\right) \quad \text{and} \quad \psi''(s) = \frac{1}{2}u''\left(\frac{\sqrt{2}}{2}s\right).$$

Therefore, since

$$c^2 u''\left(\frac{\sqrt{2}}{2}s\right) = c_0^2\left(2u\left(\frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2}\right) + 2u\left(\frac{\sqrt{2}}{2}s - \frac{\sqrt{2}}{2}\right) - 4u\left(\frac{\sqrt{2}}{2}s\right)\right) - K\sin\left(u\left(\frac{\sqrt{2}}{2}s\right)\right),$$

$$c^2 u''\left(\frac{\sqrt{2}}{2}s\right) = c_0^2\left(2u\left(\frac{\sqrt{2}}{2}(s+1)\right) + 2u\left(\frac{\sqrt{2}}{2}(s-1)\right) - 4u\left(\frac{\sqrt{2}}{2}s\right)\right) - K\sin\left(u\left(\frac{\sqrt{2}}{2}s\right)\right)$$

and

$$u\left(\frac{\sqrt{2}}{2}(s+1)\right) = \psi(s+1), \qquad u\left(\frac{\sqrt{2}}{2}(s-1)\right) = \psi(s-1),$$

after the change of variables, we arrive at the equation

$$2c^2\psi''(s) = c_0^2(2\psi(s+1) + 2\psi(s-1) - 4\psi(s)) - K\sin(\psi(s)),$$

$$c^2\psi''(s) = c_0^2(\psi(s+1) + \psi(s-1) - 2\psi(s)) - \frac{K}{2}\sin(\psi(s)).$$

Thus, for the angles $\varphi = \frac{\pi k}{2}$ and $\varphi = \frac{\pi}{4} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$, we obtain the equation studied in [10], i.e., the case of one-dimensional lattice.

Variational Statement of the Problem

Equation (3) is associated with the functional

$$J(u) := \int_{-\infty}^{+\infty} \left[\frac{c^2}{2}(u'(s))^2 - \frac{c_0^2}{2}(u(s+\cos\varphi) - u(s))^2 - \frac{c_0^2}{2}(u(s+\sin\varphi) - u(s))^2 + K(1+\cos(u(s))) \right] ds, \tag{5}$$

defined in the Hilbert space

$$X := \{u \in H^1_{\mathrm{loc}}(\mathbb{R}) : u' \in L^2(\mathbb{R})\}$$

with scalar product

$$(u, v)_X = u(0)v(0) + \int_{\mathbb{R}} u'(\tau)v'(\tau)\, d\tau.$$

Denote $\mathcal{M}_{-\pi,\pi} = \{u \in X : u(-\infty) = -\pi, u(+\infty) = \pi\}$.

Let $v_0 : \mathbb{R} \to [-\pi; \pi]$ be a monotone function in $C^\infty(\mathbb{R})$ such that $v_0(s) = -\pi$ for $s < -1$ and $v_0(s) = \pi$ for $s > 1$. We now define a functional $\Psi : H^1(\mathbb{R}) \to \mathbb{R}$ as

$$\Psi(v) := J(v_0 + v).$$

It is easy to see that $\Psi(v) < \infty$ for all $v \in H^1(\mathbb{R})$. On the contrary, the minimum point u of the functional J on $\mathcal{M}_{-\pi,\pi}$ can be written in the form $u = v_0 + v$ for some $v \in H^1(\mathbb{R})$ (see [10]). Moreover, the functional Ψ is continuously differentiable on $H^1(\mathbb{R})$.

Lemma 1. *Let u be a critical point of the functional Ψ and let*

$$u = v_0 + v \in \mathcal{M}_{-\pi,\pi} \subset X.$$

Then $u \in C^2(\mathbb{R})$ is a solution of Eq. (3) satisfying conditions (4).

Proof. Let $v \in H^1(\mathbb{R})$ be a critical point of the functional Ψ. Then $\langle \Psi'(v), h \rangle = 0$ for any $h \in H^1(\mathbb{R})$:

$$0 = \langle \Psi'(v), h \rangle = \langle J'(u), h \rangle$$

$$= \int_{-\infty}^{+\infty} [c^2 u'(s)h'(s) + c_0^2 (u(s+\cos\varphi) + u(s-\cos\varphi) - 2u(s))h(s)$$

$$+ c_0^2 (u(s+\sin\varphi) + u(s-\sin\varphi) - 2u(s))h(s) - K\sin(u(s))h(s)]\,ds\,.$$

This means that u satisfies Eq. (3) in a sense of generalized functions (weak solution). Recall that, according to the theorem on embedding, $X \subset C_b(\mathbb{R})$, where $C_b(\mathbb{R})$ is the space of bounded and continuous functions on $\mathbb{R}$. Therefore, $u \in C_b(\mathbb{R})$. Thus, the right-hand side of Eq. (3) is a continuous function. Hence, we conclude that u'' is a continuous function and, thus, $u \in C^2(\mathbb{R})$ is a solution of Eq. (3) in the ordinary sense.

For the sake of simplicity, we denote

$$(Au)(s) := u(s+\cos\varphi) - u(s), \qquad (Bu)(s) := u(s+\sin\varphi) - u(s).$$

Then, according to Lemma 3.1 in [2],

$$\|A(s)\|_{L^2(\mathbb{R})} \le |\cos\varphi| \|u'\|_{L^2(\mathbb{R})}, \qquad u \in X,$$

$$\|B(s)\|_{L^2(\mathbb{R})} \le |\sin\varphi| \|u'\|_{L^2(\mathbb{R})}, \qquad u \in X,$$

i.e.,

$$\begin{aligned} &\int_{-\infty}^{+\infty} |Au(s)|^2\,ds \le \cos^2\varphi \int_{-\infty}^{+\infty} |u'(s)|^2\,ds, \qquad u \in X, \\ &\int_{-\infty}^{+\infty} |Bu(s)|^2\,ds \le \sin^2\varphi \int_{-\infty}^{+\infty} |u'(s)|^2\,ds, \qquad u \in X. \end{aligned} \tag{6}$$

This means that

$$\int_{-\infty}^{+\infty} \left[\frac{c^2 - c_0^2}{2} (u'(s))^2 + K(1+\cos(u(s))) \right] ds$$

$$\le J(u) \le \int_{-\infty}^{+\infty} \left[\frac{c^2}{2} (u'(s))^2 + K(1+\cos(u(s))) \right] ds, \qquad u \in X.$$

Denote

$$I_\gamma(u) := \int_{-\infty}^{+\infty} [\gamma(u'(s))^2 + K(1+\cos(u(s)))]\,ds,$$

where $\gamma > 0$. Thus, according to Lemma 3.1 in [10], the functional I_γ attains its minimum on $\mathcal{M}_{-\pi,\pi}$ and, moreover,

$$\min_{u \in \mathcal{M}_{-\pi,\pi}} I_\gamma(u) = \vartheta := 2\sqrt{\gamma K} \int_{-\pi}^{\pi} \sqrt{1+\cos(\xi)}\, d\xi. \tag{7}$$

Furthermore, for the same ϑ, we get

$$\inf_{T>0} \inf \left\{ \int_{-T}^{T} [\gamma(u'(s))^2 + K(1+\cos(u(s)))] ds \ : \ u \in H^1(-T,T), \quad u(-T) = -\pi, \quad u(T) = \pi \right\} = \vartheta. \tag{8}$$

As a direct corollary from Lemma 1, estimating the integral in (7), we obtain

$$8\sqrt{(c^2 - c_0^2)K} \le \inf_{u \in \mathcal{M}_{-\pi,\pi}} J(u) \le 8c\sqrt{K}. \tag{9}$$

Thus, it follows from inequalities (9) and $1+\cos(u) \ge 0$ that

$$\frac{c^2 - c_0^2}{2} \|u_0'\|^2_{L^2(\mathbb{R})} \le J(u_0) \le 8c\sqrt{K}. \tag{10}$$

By using inequalities (9), we can easily prove the following lemma (see [10]):

Lemma 2. *Let* $c^2 > c_0^2$. *Then the point of global minimum* u_0 *of the functional* J *on* $\mathcal{M}_{-\pi,\pi}$ *satisfies the inequality*

$$\|u_0\|_{L^\infty(\mathbb{R})} < (2k+3)\pi,$$

where

$$k := \max\left\{ \kappa \in \mathbb{N}_0 : (2\kappa+1) \le \sqrt{\frac{c^2}{c^2 - c_0^2}} \right\}.$$

Main Result

We now introduce the truncated version of the functional J for the parameter $T > 1$ and $\eta \in \mathbb{R}$:

$$J_T(u; \eta) := \int_{-1/2}^{1/2} \int_{\eta - T + 1/2 + \tau}^{\eta + T - 1/2 + \tau} \frac{c^2}{2} [u'(s)]^2 \, ds\, d\tau$$

$$- \int_{\eta-T+1/2}^{\eta+T-1/2} \frac{c_0^2}{2} \left[u\left(s + \cos\varphi + \frac{1}{2} \right) - u\left(s - \frac{1}{2} \right) \right]^2 ds$$

$$- \int_{\eta-T+1/2}^{\eta+T-1/2} \frac{c_0^2}{2} \left[u\left(s + \sin\varphi + \frac{1}{2} \right) - u\left(s - \frac{1}{2} \right) \right]^2 ds$$

$$+ \int_{\eta-T+1/2}^{\eta+T-1/2} K[1 + \cos(u(s))] ds .$$

In order to prove the main result, we need the discrete version of the concentration-compactness principle (see [10] and [11] for the continuous case).

Lemma 3. *Let* $c^2 > c_0^2$ *and let* $\inf J(u)|_{\mathcal{M}_{-\pi,\pi}} \le \theta < \infty$. *Then any sequence* $(u_n) \subset \mathcal{M}_{-\pi,\pi}$ *such that*

$$\lim_{n\to\infty} J(u_n) = \theta, \tag{11}$$

contains a subsequence (u_n) *(for which we preserve the same notation) satisfying one of the following three conditions:*

(i) "concentration": there exists a sequence $(\eta_n) \subset \mathbb{R}$ *such that, for any* $\varepsilon > 0$, *one can find* $T_0 > 0$ *such that, for all* $T > T_0$,

$$J(u_n) - J_T(u_n; \eta_n) < \varepsilon \qquad \forall n \in \mathbb{N};$$

(ii) "vanishing": for all $T > 0$, *the relation*

$$\lim_{n\to\infty} \sup_{\eta\in\mathbb{R}} J_T(u_n; \eta) = 0 \tag{12}$$

is true;

(iii) "splitting": there exists $\varepsilon_1 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_1$, *one can find sequences* $f_n,\ g_n \in X$ *such that*

$$|u_n - (f_n + g_n - \pi)| \le \varepsilon,$$

$$|J(u_n) - (J(f_n) + J(g_n))| \le \varepsilon,$$

$$\lim_{n\to\infty} \operatorname{dist}(\operatorname{supp}(f_n'), \operatorname{supp}(g_n')) = \infty,$$

$$\lim_{n\to\infty} J(f_n) = \alpha, \qquad \lim_{n\to\infty} J(g_n) = \beta$$

for some $0 < \alpha$, $\beta < \theta$ *(in the first inequality,* π *is required to guarantee that* $J(f_n) < \infty$ *and* $J(g_n) < \infty$*).*

The main result of the present paper can be formulated as follows:

Theorem 1. *Let* $c^2 > \frac{9}{8}c_0^2$. *Then the point of minimum* $u_0 \in C^2(\mathbb{R})$ *of the functional* J *exists on* $\mathcal{M}_{-\pi,\pi} \subset X$.

According to Lemma 1, this point is a solution of Eq. (3) satisfying conditions (4).

The proof of this theorem is reduced to the exclusion of the last two cases (ii) and (iii) of the concentration-compactness principle.

The following two lemmas exclude the possibility of vanishing and splitting for the minimizing sequence of the functional J.

Lemma 4. *Let* $c^2 > \frac{9}{8}c_0^2$ *and let* $(u_n) \subset \mathcal{M}_{-\pi,\pi}$ *be the minimizing sequence of the functional* J. *Then condition (ii) is not satisfied.*

Lemma 5. *Let* $c^2 > \frac{9}{8}c_0^2$ *and let* $(u_n) \subset \mathcal{M}_{-\pi,\pi}$ *be a minimizing sequence of the functional* J. *Then condition (iii) is not satisfied.*

Lemmas 4 and 5 are proved in exactly the same way as Lemmas 5.1 and 5.2 in [10].

Proof of Theorem 1. Inequalities (9) mean that the functional J is bounded from below on X. Let $(u_n) \subset \mathcal{M}_{-\pi,\pi}$ be a minimizing sequence of the functional J. According to Lemma 3, for $(u_n) \subset \mathcal{M}_{-\pi,\pi}$, one of the three conditions (i)–(iii) is satisfied. It follows from Lemmas 4 and 5 that condition (**i**) is satisfied. Thus, for fixed $\varepsilon > 0$, we can choose a sequence $(\eta_n) \subset \mathbb{R}$ and $T_0 > 0$ such that

$$J(u_n) - J_T(u_n;\eta_n) < \varepsilon \qquad \forall n \in \mathbb{N}.$$

We denote $v_n(s) := u_n(\eta_n + s)$. The sequence (v_n) is bounded in the space X because

$$\|v_n'\|_{L^2(\mathbb{R})} = \|u_n'\|_{L^2(\mathbb{R})} \le \frac{2}{c^2 - c_0^2} J(u_n)$$

and $|v(0)| < 3\pi$ according to estimate (10) and Lemma 2. Since X is a Hilbert space, there exists a weakly convergent subsequence [for which we preserve the same notation (v_n)]. On the segment $[-T_0, T_0]$, the weak convergence of (v_n) is equivalent to the strong convergence in $L^2(-T_0, T_0)$ and $C^0[-T_0, T_0]$ to a certain limit u. Thus, for all $n > N$ with sufficiently large N, we find

$$\left| \int_{-T_0}^{T_0 - 1} [(Av_n(s))^2 - (Au_n(s))^2]\, ds \right| < \varepsilon,$$

$$\left| \int_{-T_0}^{T_0 - 1} [(Bv_n(s))^2 - (Bu_n(s))^2]\, ds \right| < \varepsilon$$

and

$$\left|\int_{-T_0}^{T_0}\left[\cos(v_n(s))-\cos(u_n(s))\right]ds\right|<\varepsilon .$$

Since the weak convergence means that

$$\|u'\|^2_{L^2(-T_0,T_0)}\le \lim_{n\to\infty}\inf_{n\in\mathbb{N}}\|v_n'\|^2_{L^2(-T_0,T_0)},$$

we conclude that

$$J_{T_0}(u)\le \lim_{n\to\infty}\inf_{n\in\mathbb{N}} J_{T_0}(v_n;0).$$

We choose an arbitrary monotone sequence $T_k\to\infty$ with $k\in\mathbb{N}_0$ and assume that u is already defined as the uniform limit of the sequence (v_n) on the segment $[-T_k,T_k]$. Since (v_n) is still bounded in X, we can again choose a subsequence [for which we preserve the same notation (v_n)] uniformly convergent in $C^0[-T_{k+1},T_{k+1}]$ to a certain limit $\tilde u$, which coincides with u on $[-T_k,T_k]$ (by construction).

Hence, the function u on $\mathbb{R}$ satisfies conditions (4) also with the constant $C=C(c,c_0,K)$

$$J(u)=\lim_{T\to\infty}J_T(u,0)\le \lim_{T\to\infty}\lim_{n\to\infty}\inf_{n\in\mathbb{N}}J_T(v_n,0)$$

$$\le \lim_{T\to\infty}\lim_{n\to\infty}\inf_{n\in\mathbb{N}}J(v_n)+C\varepsilon\le \lim_{n\to\infty}J(u_n)+C\varepsilon$$

and, in particular, $u'\in L^2(\mathbb{R})$. Thus, $u\in\mathcal{M}_{-\pi,\pi}$. In view of the arbitrariness of ε, it follows from the last inequality that

$$J(u)\le \lim_{n\to\infty}J(u_n).$$

This means that u is the point of minimum for the functional J on $\mathcal{M}_{-\pi,\pi}$.

Thus, we have established the conditions for the existence of heteroclinic traveling waves in a system of linearly coupled nonlinear oscillators with potential $V(r)=K(1-\cos r)$ on the two-dimensional lattice. In the nearest future, we plan to obtain the conditions for the existence of homoclinic and periodic traveling waves in systems of this kind.

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